Assignment 8

This homework is due *Thursday* Oct 31.

There are total 36 points in this assignment. 30 points is considered 100%. If you go over 30 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 4.1 in Bartle–Sherbert.

(1) (Theorem 4.1.2) Let $A \subseteq \mathbb{R}$. Prove that
  (a) [2pt] If a number $c \in \mathbb{R}$ is a cluster point of A, then there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.
  (b) [2pt] If there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$, then c is a cluster point of A.

(2) (Modified 4.1.1) In each case below, find a number $\delta > 0$ such that the corresponding inequality holds for all x such that $0 < |x - c| < \delta$. Give a *specific number* as your answer, for example $\delta = 0.0001$, or $\delta = 2.5$, or $\delta = 3/14348$, etc. (Not necessarily the largest possible.)
  (a) [2pt] $|x^3 - 1| < 1/2$, $c = 1$. (*Hint:* $x^3 - 1 = (x-1)(x^2 + x + 1)$.)
  (b) [2pt] $|x^3 - 1| < 10^{-3}$, $c = 1$.
  (c) [2pt] $|x^3 - 1| < \frac{1}{10^{-3}}$, $c = 1$.
  (d) [3pt] $|x^2 \sin x^3 - 0| < 0.00001$, $c = 0$.

(3) Reminder. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f has limit $L \in \mathbb{R}$ at c if
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.
  Below you can find (erroneous!) "definitions" of a limit of a function. In each case describe, exactly which functions "have limit L at c" according to that "definition".
  (a) [3pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
  $\forall \varepsilon > 0\ \forall \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.
  (b) [3pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
  $\exists \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.
  (c) [3pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
  $\exists \delta > 0\ \forall \varepsilon > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.

(4) (Modified 4.1.9) *Use ε–δ definition of limit* to show that
  (a) [2pt] $\lim\limits_{x \to 2} \frac{1}{1-x} = -1$,
  (b) [2pt] $\lim\limits_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$.

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(5) (4.1.11) Show that the following limits do not exist:

(a) [2pt] $\lim_{x \to 0} (x + \operatorname{sgn} x)$,

(b) [2pt] $\lim_{x \to 0} \sin(1/x^2)$.

(6) (Exercise 4.1.14) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) [3pt] Show that f has limit at $x = 0$ (*Hint*: you can use sequential criterion and squeeze theorem).

(b) [3pt] Prove that if $c \neq 0$, then f does not have limit at c. (*Hint*: you can use sequential criterion.)